CMG in a spinning satellite opposes external torques on the spacecraft. 2) The usual disturbance torques (i.e., gravity-gradient and aerodynamic torques) acting on a spinning satellite held in a prescribed attitude are either constant or periodic. Of these only a periodic applied torque of frequency ω_0 would cause a true bias momentum. 3) The y and z axis response to an impulsive torque on either axis is oscillatory.

III. An Example

The concepts expressed above are illustrated by an example. Consider a satellite which is in circular orbit and kept in alignment with the following coordinate system: x axis perpendicular to the orbital plane, and positively oriented at an acute angle relative to the ecliptic North Pole, y axis in the direction of the velocity vector, and z axis completes the orthogonal, right triad. The spacecraft is assumed to possess a nondiagonal inertia tensor and considered to be subject to a constant gravity-gradient torque. The solution to Eq. (5) for this important special case is given by:

$$H_x(t) = H_x(0+) - 3\omega_0^2 I_{yz} t$$
$$H_y(t) = T_z/\omega_0 + [H_y(0+) - T_z/\omega_0]\cos\omega_0 t + [H_z(0+) + T_y/\omega_0]\sin\omega_0 t \quad (8)$$
$$H_z(t) = -T_y/\omega_0 + [H_z(0+) + T_y/\omega_0]\cos\omega_0 t - [H_y(0+) - T_z/\omega_0]\sin\omega_0 t$$

where: $T_y = 4\omega_0^2 I_{xz}$, $T_z = -\omega_0^2 I_{xy}$, ω_0 = the orbital rate and I_{ij}; $i, j = x, y, z$ are the appropriate elements of the inertia tensor.

Note that: 1) the oscillatory terms in the expressions for $H_y(t)$ and $H_z(t)$ can be eliminated by a proper choice of $H_y(0+)$ and $H_z(0+)$; 2) an error in the choice of $H_y(0+)$ and $H_z(0+)$ will result only in a small oscillatory momentum requirement on the CMGs; 3) the initial time could have been chosen arbitrarily, i.e., it is unrelated to orbital position.

To illustrate the magnitude of these effects, consider a spacecraft similar to that of the first Saturn V Workshop of NASA's Skylab Program. The geometry is essentially cylindrical and for this example, the axis of symmetry, x, is perpendicular to the orbital plane and the geometric y axis is along the velocity vector. The orbital altitude is taken to be 235 naut miles. For this case‡: $\omega_0 = 1.0841 \times 10^{-3}$ (sec)$^{-1}$, $I_{yz} = -19{,}871$ slugft^2, $I_{xz} = 227{,}330$ slugft^2, and $I_{xy} = 1{,}412$ slugft^2. For these values: $T_y/\omega_0 = 986$ ftlbsec, $T_z/\omega_0 = -1.5$ ftlbsec, and the increase in bias momentum per orbit for $H_x(t)$ is 406 ftlbsec.

It is of interest to compare these results to the CMG momentum variation in a corresponding inertial mode. If the same spacecraft at the same orbital altitude is now held in the following orientation: x axis (the geometric axis of cylindrical symmetry) perpendicular to the orbital plane, and positively oriented at an acute angle relative to the ecliptic North Pole, z axis along the intersection of the orbital plane with the orbital noon meridian plane and positively directed towards the geocenter, and y axis completing the orthogonal, right triad, then

$$H_x(t) = H_x(t_0) - 42\cos(2\omega_0 t) + 32\sin(2\omega_0 t) \text{ ftlbsec}$$
$$H_y(t) = H_y(t_0) + 2{,}325\Delta t/\tau + 185\sin(2\omega_0 t) \text{ ftlbsec} \quad (9)$$
$$H_z(t) = H_z(t_0) - 15\Delta t/\tau + 185\cos(2\omega_0 t) \text{ ftlbsec}$$

where $\Delta t = (t - t_0)$ and τ = the orbital period = $2\pi/\omega_0$. In this case, almost all the bias momentum accumulates on the y axis and it is seen that the momentum that must be dumped per orbit is 2325 ftlbsec, almost six times larger than that for the spinning configuration.

‡ These values are illustrative only and should not be considered the actual values of the Skylab Workshop.

IV. Concluding Comments

The results presented here are independent of the number, type and mounting geometry of the CMGs. For a given CMG configuration, $\dot{\mathbf{H}}$ and $\mathbf{H}$ must be chosen to satisfy Eq. (5) without violating any constraints on the gimbal angles and rates.[2,3]

The aerodynamic torque acting on a satellite in an Earth-pointing attitude would be constant if the atmospheric density were constant. However, due to the diurnal bulge effect in the atmosphere, the aerodynamic torque term, if included on the right side of Eq. (5), would contain a sinusoidal term of frequency ω_0. For example, a conservative estimate of the bias momentum accumulation about the y and z body axes of the First Skylab Workshop due to this component of the aerodynamic torque is approximately 150 ftlbsec/axis/orbit.

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Determination of Heat-Transfer Rates from Transient Surface Temperature Measurements

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Nomenclature

k = thermal conductivity
N = number of equal time intervals
q = theoretical heat-transfer rate per unit area (heat flux)
Q = theoretical heat transferred per unit area to the semi-infinite solid
T = temperature change in the solid
T_s = stepwise change in surface temperature
α = thermal diffusivity
ρc = product of density and specific heat
ϕ = Kirchhoff variable

Subscripts

i,j = quantity at $x = i\Delta x$ and $t = j\Delta t$
n = quantity computed by a numerical method
o = properties evaluated at initial uniform temperature of solid
v = quantity obtained numerically considering all properties variable
α = quantity obtained numerically considering α independent of ϕ

A WELL-KNOWN method for determining heat-transfer rates in short flow duration facilities such as shock tubes and shock tunnels is that in which heat-transfer rates are inferred from the measurement of the surface temperature history of a solid exposed to the flow. Heat conduction in the substrate of the gage consisting of a surface-temperature sensor and the solid, which is of sufficient thickness to behave as a semi-infinite medium during the testing time, is treated by

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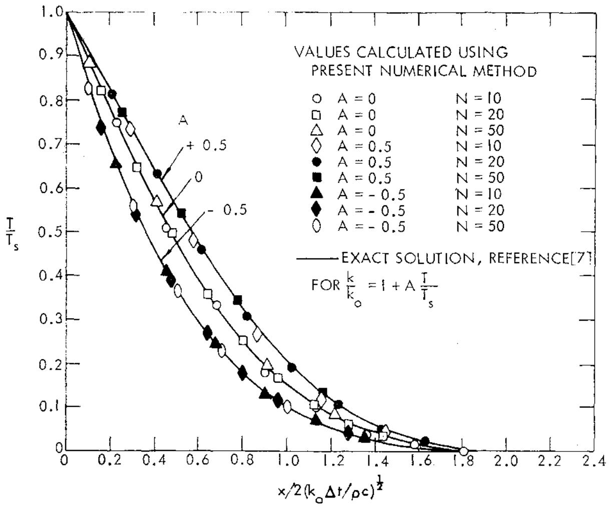

Fig. 1 Comparison of temperature profiles of Yang's solution with ρc constant and those predicted by the present numerical method.

one-dimensional heat conduction theory with the measured surface temperature serving as a boundary condition to yield the surface heat flux. This method was originally developed by Vidal,[1] who used a thin metallic film as a resistance thermometer to measure the surface temperature. It has recently been extended to the use of thin-film thermocouples to measure the surface temperature (e.g., Ref. 2). When substrate thermal properties are treated as constant, heat flux values can be accurately determined for arbitrary surface temperature histories by the numerical method of Ref. 3. This method yields the following expression which results from Eq. (4) of Ref. 3 written in terms of $T(0,t)$:

$$q_n(t_m) = \frac{2(k\rho c)_0^{1/2}}{\pi^{1/2}} \sum_{I=1}^{m} \frac{T(0,t_I) - T(0,t_{I-1})}{(t_m - t_I)^{1/2} - (t_m - t_{I-1})^{1/2}} \quad (1)$$

where t_I is the time at the end of the Ith of m time intervals.

The assumption of constant substrate properties is appropriate when substrate temperature change is small or where substrate thermal properties are nearly independent of temperature. Under certain heating conditions, however, the substrate temperature rises considerably. Thus it appears that the assumption of constant substrate properties could introduce significant error into the determination of heat flux. This note presents a method that accurately incorporates the variation of substrate properties into the computation of heat flux for arbitrary surface temperature histories.

Treating the substrate as a homogeneous isotropic solid and introducing the Kirchhoff transformation[4]

$$\phi = \int_0^T \frac{k}{k_o} dT$$

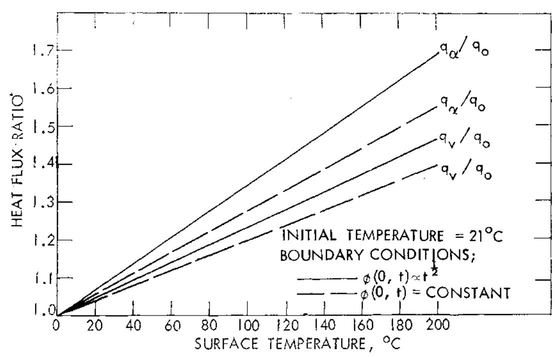

Fig. 2 Heat flux ratios for Code 7740 pyrex determined using the present numerical method and properties from Table II, Ref. 8.

Table 1 Ratios of heat flux numerically determined by equation noted to exact heat flux with $\alpha(\phi)$ = const and k and ρc variable

Boundary Condition:	$\phi \propto t^{1/2}$	$[q(t)$ = const]	ϕ = const	$[q(t) \propto t^{-1/2}]$
	$q_n(t)/q(t)$		$q_n(t)/q(t)$	
	Eq. (1)[a]	Eq. (7)	Eq. (1)[a]	Eq. (7)
N = 10	1.004	1.002	1.026	1.026
20	1.001	1.001	1.013	1.013
50	1.000	1.000	1.005	1.005

[a] With T replaced by ϕ.

the governing equation for one-dimensional unsteady heat conduction becomes

$$\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{1}{\alpha(\phi)} \frac{\partial\phi(x,t)}{\partial t} \quad (2)$$

The initial and boundary conditions for the present problem are $\phi(x,0) = 0$, $\phi(\infty,t) = 0$ and

$$\phi(0,t) = \int_0^{T(0,t)} \frac{k}{k_o} dT \quad (3)$$

where the surface temperature $T(0,t)$ is an arbitrary function. The total heat transferred per unit area to the solid up to time t is

$$Q(t) = \int_0^\infty \int_0^{T(x,t)} \rho c dT dx \quad (4)$$

and the surface heat flux can be expressed as

$$q(t) = dQ(t)/dt \quad (5)$$

In the preceding equations all thermal properties are considered to be temperature dependent. Thus Eq. (2) is generally nonlinear. Therefore an analytical solution for $q(t)$ cannot be readily obtained for arbitrary $T(0,t)$. However, one may use numerical methods to obtain the desired solution. The numerical method employed here involves a minor modification of the well-known explicit finite difference technique commonly applied to linear parabolic partial differential equations.[5] Here, α is assumed locally constant over a fixed distance Δx for a fixed time interval Δt and Eq. (2) is solved in approximate form to yield

$$\phi_{i,j+1} = \epsilon_{i,j}(\phi_{i+1,j} + \phi_{i-1,j}) + \phi_{i,j}(1 - 2\epsilon_{i,j}) \quad (6)$$

where $\phi_{i,j} = \phi(i\Delta x, j\Delta t)$, $i = 1, 2, 3, \ldots$, and $j = 1, 2, 3, \ldots N$. Also, $\epsilon_{i,j} = \alpha_{i,j}\Delta t/(\Delta x)^2 \leq \frac{1}{2}$ for stability of the solution, where $\alpha_{i,j} = \alpha(\phi_{i,j})$. In application of Eq. (6) to various x positions and successive time intervals, $\epsilon_{i,j}$ varies as α varies since Δx and Δt are fixed. Equation (6) can then be applied to numerically determine the ϕ (and therefore T) profiles in the substrate at times $j\Delta t$ subject to the condition of initially uniform temperature and the boundary condition $\phi(0,t)$ obtained from Eq. (3) and the prescribed $T(0,t)$. Equation (4) can be written in finite difference form as

$$Q_n(j\Delta t) = T_{0,j}(\rho c)_{0,j} \frac{\Delta x}{2} + \sum_{i=1}^{l=j} T_{i,j}(\rho c)_{i,j}\Delta x$$

Table 2 Ratios of heat flux determined by present numerical method to exact heat flux based on Yang's solution with ρc constant

	$q_n(t)/q(t)$		
A	N = 10	N = 20	N = 50
+0.5	1.031	1.016	1.007
0	1.027	1.013	1.005
−0.5	1.006	1.002	1.001

where

$$(\rho c)_{i,j} = \left\{\int_0^{T_{i,j}} \rho c dT\right\}/T_{i,j}$$

and the heat flux from Eq. (5) can be written in numerical form as

$$q_n(j\Delta t) = \{Q_n[(j+1)\Delta t] - Q_n[(j-1)\Delta t]\}/2\Delta t \qquad (7)$$

Thus by this method the heat flux can be numerically evaluated for arbitrary $T(0,t)$ and any prescribed variation of thermal properties with temperature. Additional details and computational techniques are discussed in Ref. 6.

To illustrate the accuracy of the present method, numerical solutions will be presented for two special cases of property variation; α independent of ϕ with variable ρc and k and constant ρc with variable k and α. When α is independent of ϕ, Eq. (2) is linear and of the same form as the corresponding equation obtained in terms of T when properties are assumed constant. Thus Eq. (1) obtained for constant properties is applicable when $\alpha(\phi)$ = constant if T in that equation is replaced by ϕ. Table 1 presents for $\alpha(\phi)$ = constant a comparison of theoretical solutions and numerical solutions obtained through applications of Eq. (1) and Eq. (7) for two boundary conditions for which exact solutions for heat flux can be readily obtained. For each boundary condition it is noted that the error associated with each numerical method is small, and decreases with increasing N.

The case of ρc constant with variable k and α was considered, since for this case an exact solution to the governing equations obtained by Yang[7] for a semi-infinite solid and a stepwise change in surface temperature is available for comparison purposes. Figure 1 presents a comparison of theoretical and numerically determined temperature profiles in terms of Yang's similarity variables T/T_s and $x/2(k_o\Delta t/\rho c)^{1/2}$ for Yang's case of linear variation of k with temperature as noted in the figure. Since for the present case ρc is assumed constant, α also varies linearly with temperature. Table 2 presents a comparison of numerically determined and exact heat-transfer rates based on Yang's solution. In the computations for Fig. 1 and Table 2, α varied by about $\pm$ 50%. It is noted from Fig. 1 and Table 2 that the agreement between Yang's solution and the present solution is very good and increases with N. The results in Tables 1 and 2 and Fig. 1 are considered representative of the accuracy in the computation of heat-transfer rates that can be obtained for arbitrary $T(0,t)$ and variable substrate properties using the present method. If α for the heat-flux gage substrate is independent of ϕ or if it is established by the present method that the influence of α is small, Eq. (1) with T replaced by ϕ is preferred over the present method since numerical evaluation of Eq. (1) requires less computer time.

Hartunian and Varwig[8] analyzed the influence of variable substrate properties on heat flux inferred from surface temperature measurements made by thin-film heat-flux gages. Code 7740 pyrex, one of the substrate materials considered in Ref. 8, is a commonly used substrate for thin-film heat-flux gages. Hence it is of interest to analyze by the present method the influence of variable properties in this material on heat-transfer rates inferred from surface temperature histories. Figure 2 shows results of computations by the present method for two boundary conditions, $\phi \propto t^{1/2}$ and ϕ = const, for 7740 pyrex using property values from Table II, Ref. 8. Here, q_v/q_o is the ratio of the heat flux computed considering all properties variable to the heat flux computed assuming constant properties, whereas q_α/q_o is the ratio of heat flux computed assuming α independent of ϕ to the heat flux computed assuming constant properties. It is noted in Fig. 2 first that the boundary condition imposed has an influence on the ratios and second, that it is important to consider for this material the variation in α as well as k when temperature change is large. The latter is evidenced by the departure of q_α/q_o from q_v/q_o as both depart from unity with increasing temperature for both boundary conditions. The curves in Fig. 2 indicate that for 7740 pyrex neither the assumption of constant properties nor the assumption of α independent of ϕ is satisfactory at higher values of temperature change. The results in Fig. 2 differ somewhat from those obtained by Hartunian and Varwig for the same material using an approximate theoretical solution of the governing equations and boundary conditions equivalent to those in Fig. 2. For example, Hartunian and Varwig list values of 1.45 and 1.44 for q_α/q_o and q_v/q_o at T = 150°C for both boundary conditions. However, the present method yields corresponding values of 1.52 and 1.35 for the parabolic boundary condition and 1.41 and 1.30 for the step-boundary condition. In view of the verification obtained for the present method it is concluded that for 7740 pyrex the influence of both the boundary condition and variable diffusivity are each more significant than suggested by Hartunian and Varwig.

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Center of Flexure of Arbitrary Cross Sections

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In a previous paper[1] the general formula for the position of the center of flexure has been developed in the form

$$y_0 = \frac{1}{F}\iint\left\{y\frac{\partial\psi}{\partial y} - x\frac{\partial\psi}{\partial x} - F\frac{yx^2}{2I}\right\}dxdy \qquad (1)$$

where $Oxyz$ = orthogonal coordinate system; x,y = principal axes of the cross section; F = vertical load parallel to the x axis; $I = \iint x^2dxdy$ = second moment of area about y axis; $\psi(x,y)$ = stress function defined by

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial x^2} = \frac{\nu}{1+\nu}\frac{Fy}{I}$$

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